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ABSTRACT

This paper reports on a series of four studies carried out over a period of four years. These related studies were clinical and qualitative as they investigated middle and high school students' development of geometric thought, particularly as it related to three- dimensional visualization. The studies were carried out in the constructivist pedagogical style. The Geometer's Sketchpad (GSP) was instrumental in all of these studies and it was this environment in which the subjects learned to increasingly develop their visualization abilities as they solved challenging geometry problems. Platonic solids, Archimedian solids, other geometric solids, and the mathematical relationships related to these solids were the objects of study. Students' van Hiele levels of geometric thinking were monitored as they constructed dynamic GSP representations of these solids to conduct geometric investigations. A fundamental finding was that GSP provided opportunities to have a distinct positive affect on students' learning of three-dimensional geometry. (Author/NB)

STUDENTS' DEVELOPMENT OF THREE-DIMENSIONAL VISUALIZATION IN THE GEOMETER'S SKETCHPAD ENVIRONMENT

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This article reports a series of four studies carried out over a period of four years. These related studies were clinical and qualitative, as they investigated middle and high school students' development of geometric thought, particularly as it related to three-dimensional visualization. The studies were carried out in the constructivist pedagogical style. The Geometer's Sketchpad was instrumental in all of these studies and it was this environment in which the subjects learned to increasingly develop their visualization abilities as they solved challenging geometry problems. Platonic solids, Archimedian solids, other geometric solids, and the mathematical relationships related to these solids were the objects of study. Students' van Hiele levels of geometric thinking were monitored as they constructed dynamic GSP representations of these solids to conduct geometric investigations. A fundamental finding was that GSP provided opportunities to have a distinct positive affect on students' learning of three-dimensional geometry.

According to the NCTM Standards documents (1989, 2000), in high school geometry, all students should have opportunities to visualize and work with three-dimensional figures in order to develop spatial skills fundamental to everyday life and to many careers. Geometry instruction should focus increased attention on the analysis of three-dimensional figures. It should focus on the continued development of students' skills in visualization and pictorial representation of three-dimensional figures.

Research (Ben-Chaim, Lappan, & Hoaung, 1988) suggests that spatial strategies can be taught successfully to middle and high school students. Appropriate use of three-dimensional representation and computer software is of particular value in development of three-dimensional visualization. Particularly, the Geometer's Sketchpad (GSP) developed by Jackiw (1995), a dynamic geometry construction and exploration tool has become widely introduced into classrooms of mathematics and science, especially of geometry. Its effects on different fields have been topics of research. Dixon (1997) investigated its effects on eighth grade students' spatial visualization. Her research results suggest that the GSP instructional environment was more effective than the traditional instructional environment at improving students' two-dimensional visualization, but was not more effective than the traditional instructional environment at improving students' three-dimensional visualization.

Bishop (1983), in describing his previous works in spatial thinking, presents two different but similar spatial abilities:

1) The ability for interpreting figural information (IFI). This ability involves understanding the visual representations and spatial vocabulary used in

geometric work, graphs, charts, and diagrams of all types. Mathematics abounds with such forms and IFI concerns the reading, understanding, and interpreting of such information. It is an ability of content and of context, and relates particularly to the form of the stimulus material. 2) The ability for visual processing (VP). This ability involves visualization and the translation of abstract relationships and nonfigural information into visual terms. It also includes the manipulation and transformation of visual representations and visual imagery. It is an ability of process, and does not relate to the form of stimulus material presented. (p. 185)

According to Bishop, these definitions refine and extend the definitions given by McGee (1979). IFI extends the spatial orientation of McGee to include geometric and graphical conventions and refines it by emphasizing "the interpretation demanded by those representations" (p. 185). VP as well is refined and extended by emphasizing "the process aspect rather than the form of the stimulus" (p. 185). Thus, three-dimensional spatial ability includes both the ability of interpreting figural information in the context of three-dimensional objects and the ability for visual processing of these objects.

The Purpose of the Research

The purpose of the research was to investigate the effects of the GSP dynamic instructional environment on middle and high school students' three-dimensional visualization. The following questions were investigated in the studies:

1) What role can the GSP dynamic instructional environment play in the development of students' three-dimensional visualization? Is there evidence to indicate improvement in students' three-dimensional spatial ability when GSP is used to teach three-dimensional geometry?
2) What role can the GSP dynamic instructional environment play in the development of students' geometric thinking as defined by the van Hiele theory? Is there evidence to indicate improvement in students' geometric thinking when GSP is used to teach three-dimensional geometry?

Significance

This article describes a series of research studies that focus on the use of GSP in improving students' three-dimensional visualization, one of the important aspects in developing a reform-oriented geometry curriculum. More and more educators believe that the use of technology can effectively facilitate the teaching and learning of mathematics. The NCTM Standards documents (1989, 2000) emphasize the effective use of technology as one of the main feature of the reform curriculum. However, technology-related research on students' three-dimensional visualization is very limited. Increasingly, mathematics educators and teachers need quality technology-related research to "inform us of the conceptual linkages among new and old ideas and orientations and how these might be influenced by various instructional strategies and materials" (Kaput & Thompson, 1994, p. 680). It is this concern that the research studies took into consideration. They explored evidence supporting the belief that students could benefit from the use of technology. The research results will contribute to our current limited understanding of the students' development of three-dimensional visualization and geometric thought within a technology-rich environment.

Conceptual Framework

The conceptual framework of the research was the constructivist perspective that knowledge is not passively received from the teacher but actively constructed by the learners themselves, the van Hiele model of students' geometric thought, and the previous research on three-dimensional visualization. According to van Hiele (1986), students progress in geometric thinking through taxonomy of levels, and progress from one level to the next higher level is dependent on the nature of the instruction provided to students. The five van Hiele levels are Level 1: Visual (figures recognized by appearance alone), Level 2: Analysis of Properties (properties perceived, but isolated and unrelated), Level 3: Ordering/Hierarchy (relationships, implications, & class inclusions), Level 4: Deduction/Proof (deduction is meaningful, can construct proofs), and Level 5: Rigor (formal aspects of deductions, symbols manipulated). Within this framework, the van Hiele model was examined in light of the use of technology. van Hiele's research and particularly the view of levels of thought and the lack of communication among levels may need to be updated for an educational system in which technology is an integral component.

Design and Procedure

Constructivist teaching experiments were used for these investigations. Over a four-year period, twenty-four students were provided increasingly more challenging geometry units with GSP as a central feature throughout. The class as a whole was engaged in problem solving in geometry, which included GSP constructions, investigations, and dynamic experimentation. However, since we were interested in the students' detailed changes and growth in their thinking processes, we focused on looking at individuals and used a clinical approach to assess students' growth in geometric thought.

The class of twenty-four students was a part of the Partnership Academic Community (PAC) program, a collaboration between Florida International University and Miami-Dade County Public Schools. This program aims to help at-risk minority students of metro Miami area to improve their academic and behavioral achievement by placing them in a reform-oriented, technology-rich learning environment. The PAC students are from three middle schools and a high school of Miami-Dade School District. Most of them are African-Americans and Hispanics and from lower socio-

Table 1. Research Phases 1 and 2

	Phase 1	Phase 2
Grade Level	Grade 7	Grade 8
Length of Study	20 weeks at 2 hours per week	10 weeks at 3 hours per week
Subjects	Whole class of students	Whole class of students
Students' Learning Activities	Constructions of basic geometric figures, and explorations on geometric transformations, projective concepts, perspectives, and dynamic representations of geometric solids	Further explorations on dynamic representations of geometric solids, and simple reasoning challenges.

Table 2. Research Phases 3 and 4

	Phase 3	Phase 4
Grade Level	Grade 9	Grade 10
Length of Study	15 weeks at 1 hour per week	10 weeks at 7 hours per week
Subjects	Four subjects (two boys and two girls)	Four subjects (two boys and two girls)
Students' Learning Activities	Problem-solving sessions and task-based interviews	Problem-solving sessions and constructivist teaching interviews
Research Focus	Subjects' ideas and reasoning processes concerning their solutions to the problems, and their van Hiele level of thinking	Students' conceptual understanding , their ability to solve problems by modeling, and their van Hiele level of thinking

economic families. The subjects involved in the clinical studies were selected for their willingness and also for their different mathematics thinking levels.

The teaching experiments progressed through four phases, which are summarized in the following two tables:

A part of the progression from study to study is the extent to which the envelope is being pushed. For example, in studies dating back to 1997, the inquiry was on tasks directly related to the studies of the van Hieles, of basically plane geometry concepts with an easy entry into solid geometry concepts such as perspective views of solid figures. In more recent studies, the investigation of plane geometric concepts embedded in three- dimensional objects was undertaken. Another dimension of growth in inquiry was providing subjects with physical geometric objects and requesting their construction in GSP as dynamic representations on which transformations such as rotations and translations and specific components such as cross sections could be investigated.

The exploration activities/problems used in the studies were devised in ways that (a) the subjects were required to use hands-on explorations with GSP to learn new geometric ideas; (b) they were required to make conjectures based on their explorations; (c) they were required to make meaningful explanations for whatever conjectures they made; and (d) they were asked to look back to check if they had reached a complete understanding. Our goal was to use these activities to help each subject develop three-dimensional visualization and make orderly progress in the movement from the van Hiele level he or she was currently at to a higher level.

Data Collection

Detailed notes were taken when observing the teaching sessions in all four phases. Some of the sessions and all teaching interviews were videotaped. Significant parts of the videotapes were transcribed. The computer files on the disks, and the written tasks completed by the students throughout the four phases, were also collected.

Data Analysis

A constant comparative approach was used to analyze the collected data. The constant comparative method of qualitative data analysis (Glaser & Strauss, 1967; Grove, 1988) serves as a model demonstrating how systematic analysis can improve the validity of a qualitative research study. Constant comparison involves analyzing and interpreting data during and after data collection. Using this method, the transcripts and other data were analyzed during teaching sessions and after all sessions were completed. These analyses took the form of editing a set of transcripts and other data with commentaries. The commentaries depicted the story lines of the corresponding students' progression through the sessions.

Results

The results of the research suggest that GSP and the associated activities were effective in helping the students develop three-dimensional visualization and achieve conceptual understanding of geometry content.

GSP helped the students generate and sustain insight and enthusiasm in learning three-dimensional geometry.

Starting in grade 7, the subjects of these studies had positive attitudes toward the use of GSP. They found it interesting and dynamic. They found it a friendly environment in which they could play around and experiment. Using GSP to learn geometry in an experimental fashion has contributed to the subjects' enthusiasm for geometric exploration. In some sense, GSP was not considered a geometric environment so much as an artistic and creative environment by these subjects.

Periodically, the students in these studies were asked their views of using computers, particularly their views related to GSP. Consistently, they expressed their continuing enjoyment and valuing of GSP as a tool for learning. The ways that the subjects used GSP changed form over the years. Initially subjects made simple creative uses as they learned the capabilities of GSP. Then they began to make sense of problem situations; for example, making operational the definition of geometric objects (e.g., a dodecahedron). The three-dimensional uses of GSP emerged early (during the 7th grade) when they turned to the study of perspective drawing. Problems such as finding a hexagonal cross section of a cube and showing it visually moved the use of GSP to focused challenges.

The dynamic possibilities of GSP emerged during the 8th grade year, as students used hints and challenges to do transformations of three-dimensional objects and their cross sections. They expressed the positive attitudes that emerged in the new ways of experiencing the GSP environment. One of our subjects expressed it well, as follows:

> I like the ones that move...the animation...it's different from the object just staying still on the screen...It becomes easier because that knowledge from GSP is in your head and your head is working like GSP, so you can think about it, rotate it. GSP helps you really think about it, you use the program in your head, as a screen in your mind so that if you construct a figure...say a perpendicular line, you can see it in the screen (in your mind).

The dynamics held continuing opportunities for geometric exploration far beyond what we had done before, and the subjects were able to create solids that could be transformed. The solids could be rotated, and the cross section slices on it could be translated and rotated as well. Our subjects found these uses of GSP brought new challenges. They realized the implications for a deeper level of study of geometric objects. In the 9th and 10th grade years, Platonic solids and Archimedian solids became objects of study. The animation of GSP helped students to analyze the structure of these objects in an in-depth way. Some tasks were provided that challenged the subjects of these studies to make sense of the meaning of duality while finding the dual of each Platonic solid. Other tasks challenged our subjects to use slicing planes to create Archimedian solids from given Platonic solids. These tasks brought a new dimension to the GSP exploration. Likewise, tasks that requested our subjects to construct in GSP an Archimedian solid from the net of such a solid expected higher levels of visualization. Through these types of tasks, our subjects built GSP planar representations of the related polyhedra. The use of these dynamic GSP representations permitted our subjects to study the features of each solid, and for some the result was an understanding of the structure of the solid. However, possibly the greater value of these tasks was that they enticed the subjects into a greater mental involvement in the study of three-dimensional geometry. Their enjoyment of this type of study was a key, we believe, to the development of spatial thinking of our subjects. The evidence that we have, beyond what we could observe the students doing and saying was that about 50% of them reported in post-study surveys the influence that GSP had upon their spatial understanding. Some of the students' comments were:

> "I like working on the computer more because it is fun, and GSP helps me to learn geometry but at a higher level of understanding because I have all the visual I need right in front of me."

> "I learned a lot about things like the properties of solids, regularity, shapes, and 2-D dynamic constructions. The computers really helped me to see and know these things because sometimes I couldn't visualize these things that were so easily shown on GSP... The GSP program would be a great learning tool to use because the student in order to construct a figure or a shape must take the properties of that shape into consideration."

> "GSP helped us look at the objects and shapes from different points of view and in a way that helped record the information into our minds."

Thus the GSP dynamic environment effectively facilitated the affective learning, and the sense of enjoyment and ownership of our subjects. We believe this affective learning indirectly contributes to the development of spatial visualization abilities.

GSP, by providing immediate feedback, helped students efficiently confirm or correct their conjectures.

In the teaching experiments, the investigators followed the constructivist approach where the subjects were requested to learn new ideas by exploring given task situations. As the first step of the exploration, the investigator always encouraged the subjects to make initial conjectures. Looking back over the many initial conjectures the subjects made we found that a large portion of those conjectures were incorrect. Making incorrect initial conjectures was not a bad thing, and seemed to be normal in students' learning process. From the constructivist point of view, substantive learning takes place over a long period of time and occurs during periods of confusion and conflict. In the teaching sessions designed for these studies, as soon as the subjects made their initial conjectures, they were allowed (and at the same time they were very eager) to manipulate the GSP dynamic models to check their ideas. The immediate visual feedback provided by these models stimulated the subjects' strong desire to struggle against and clear their mental confusion and conflict, thus achieving improved understanding. The interview with a subject (Laura) in Phase 3 gives a good example of this aspect.

The interview was designed to compare the GSP dynamic planar representation with a static paper-and-pencil test in facilitating the subject's three-dimensional visualization. Laura was given a two-dimensional figure of a regular icosahedron. The given figure only showed its front elements (vertices, edges, and faces), with its back elements missing (Figure 1). The task was to add all missing elements to complete the figure.

Laura took a pencil-and-paper test first. To add the missing elements, she tried hard to image the spatial structure of a regular icosahedron. Even though she added some elements (e.g., one missing vertex on the upper pentagon cross section and two missing vertices on the lower pentagon cross section) correctly, she was not able to visualize the complete structure of the regular icosahedron. Figure 2 shows her drawing - an apparently incorrect conjecture.

Soon after the paper-and-pencil version of the task, Laura began the GSP test. In this case, she was given a GSP sketch of dynamic planar representation of the regular icosahedron. She rotated the planar representation by dragging the free point (on the small circle) around (see Figure 3). In the course of rotation, she had opportunity to observe the continuous change of images of the icosahedron. In other words, she experienced different orientations of the planar representation of the solid. This effectively enhanced her visualization on the solid. She quickly said, "Oh, my drawing was wrong." To add the missing elements correctly, she continued to drag the free point and moved it around, sometime fast, sometime slowly.

The investigator observed that when the subject rotated the figure, she approximated the positions of the missing elements and checked her approximation by slow-

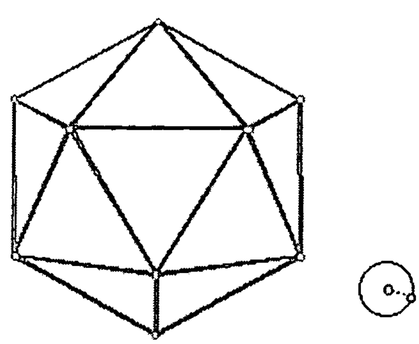


Figure 1. A figure of an icosahedron with missing elements.

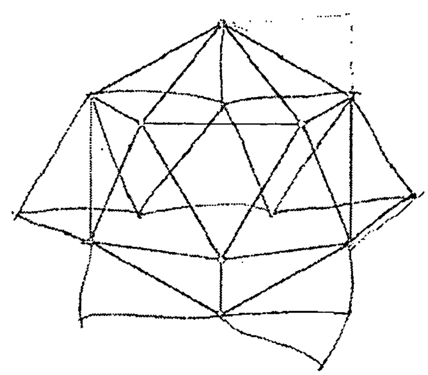


Figure 2. Laura's drawing for the missing elements.

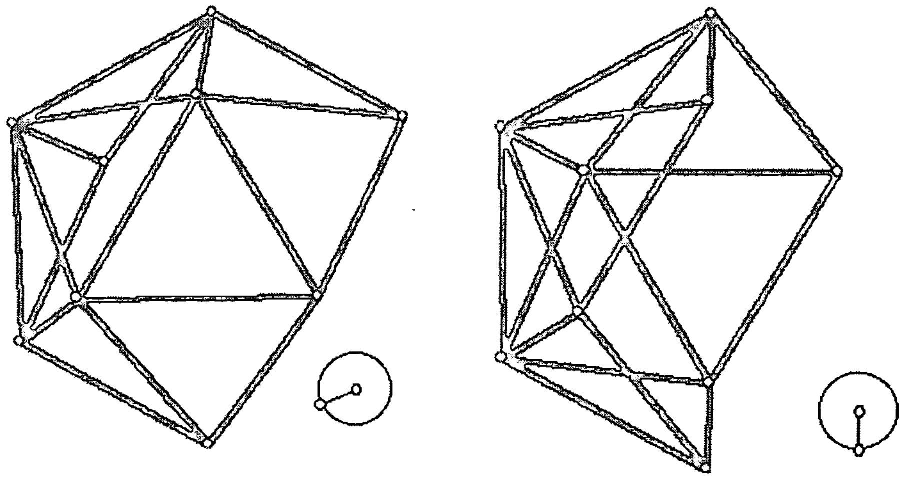


Figure 3. Two specific positions in the course of rotation.

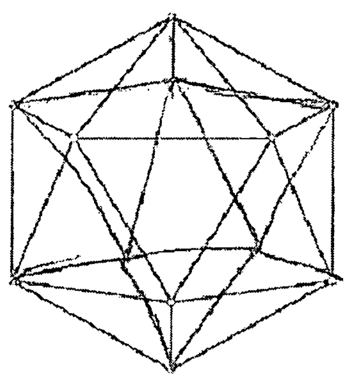


Figure 4. Laura's drawing in the GSP test.

ing down her rotation and moving the free point back and forth. In about three minutes, she finished her drawing (See Figure 4). This time, she figured out the positions of all missing vertices accurately and connected all missing edges correctly. The drawing suggests that with the feedback that GSP animation provided, Laura corrected her misconceptions and achieved a better understanding of the spatial structure of the regular icosahedron.

In the paper-and-pencil test Laura was struggling to visualize the complete spatial structure of the object, but she failed based on the limited information that the static planar figure provided. She was only able to develop an incomplete internal representation of the solid. After she finished the GSP test, the investigator had a conversation with her.

I: Look at the picture you drew on your paper-and-pencil test.

L: I know there was something wrong.

I: I would like to understand how you drew the picture like this.

L: Well, I was thinking there should be ten faces in the middle.

I: Could you try counting the faces? How would that help?

Then she began to count the faces in the middle. "Here we go. When I counted the faces in the middle, I missed these two triangles," she said, while pointing at the two most "outside" triangular faces (see Figure 2). Thus she was able to clear up her own confusion. The conversation allowed us to see the clear picture of her mental process. At first, she had some understanding of the spatial structure of the object, and so she knew there were five faces in the upper part and ten faces in the middle. In fact, she drew the missing vertices in the upper cross section correctly. However, because of her confused spatial visualization on the static planar figure she was not able to draw the missing vertices in the lower part correctly. In the GSP test, through manipulating the dynamic planar representation of the solid, she reached a better understanding of the structural properties, made progress on the logical connections, and found out what was wrong with her previous drawing. Thus, she developed a more complete mental representation of the solid.

Another example is the following task given to the subjects in the interview sessions in Phase 4. They were requested to produce and study the polyhedron represented by the net shown in Figure 5. Typical misconceptions that arose in this setting included interpreting the adjacent equilateral triangles as a parallelogram. Two of the subjects then concluded that the resulting polyhedron was a rectangular prism, and two concluded that the solid could not be formed. To help the subjects visualize the solid they were studying, they were introduced to a pre-constructed GSP sketch that allowed them to fold the net into a solid and unfold the solid back to the net by clicking the related buttons. This experience allowed the subjects to see that the triangles could fold into different planes (i.e., their conjectures were incorrect) and led them to

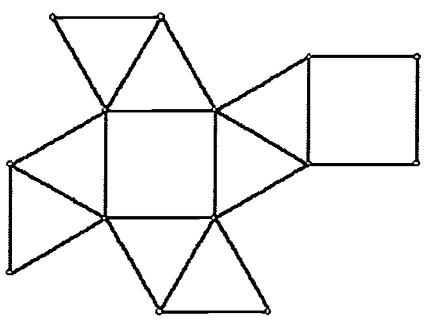


Figure 5. Task: Construct the polyhedron that this net represents.

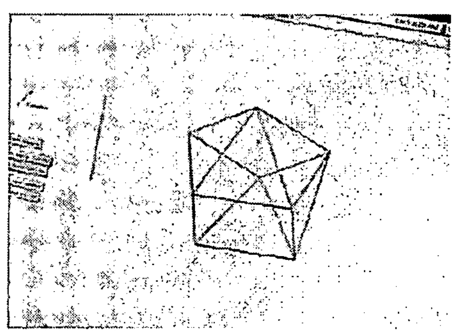


Figure 6. Manuel 's GSP construction of the polyhedron from the net.

re-analyze the net. This type of feedback mechanism became a pattern of inquiry that aided visualization. With minimal study, the subjects were able to determine the polyhedron that the net represented. One of the subjects (Manuel) was able to construct a dynamic GSP model of the solid within a few minutes because he was able to visualize the polyhedron. He rotated his GSP model and quickly indicated that this was it. The rotations enabled him to test out this construction and see if it fitted his image. This use of GSP is instructive and provides avenues for constructing meaning.

Along with the subjects' progression in the teaching experiments, we did find the subjects gradually made fewer incorrect conjectures.

The GSP dynamic environment helped the students concentrate on the logical (rather than visual) properties of three-dimensional objects.

For a two-dimensional object, if we construct its external representation which preserve all of its logical properties, then the visual properties of the representation are consistent with the logical properties of the geometric object. In this case, students can guess some properties of a geometric object based on the visual properties (using observation and measurement tools) due to the consistency. For a three-dimensional object, however, even though we construct its two-dimensional representation using the perspective approach, the visual properties of the two-dimensional representation are usually not consistent with the logical properties of the geometric object. For example, parallel perspective planar representation maintains some properties of three-dimensional objects such as parallelism and the ratio of collinear segments, but the representation does not keep some other properties of the objects such as the perpendicularity or congruency of segments and angles. Due to the influence of the approaches they applied to exploring two-dimensional objects and the limitation of their logical reasoning ability, students tend to draw conclusions based on the visual properties of the static planar representation rather than the logical properties of the three-dimensional objects. GSP can provide students with dynamic two-dimensional representations of three-dimensional objects, and adding the dynamic dimension can help students overcome the obstacle. That is, by manipulating the dynamic planar representations such as rotating them, students are able to induce logical properties of the geometric objects, because the visual properties of the external representations move to the background. In addition, students can control the motion. In this way the GSP dynamic instructional environment is valuable to develop students' internal representation (mental image) of three-dimensional objects.

As an illustration of how GSP dynamics helped our young subjects user logical properties to make and verify conjectures, we offer the following example - the problem situation shown in Figure 7, as it was presented to a subject (Mary) in Phase 3:

The problem asked for a comparison between the two segments highlighted in the two dimensional-representation of a regular icosahedron. In the paper-and-pencil test that was given to her first, Mary thought that segment US was not congruent to but longer than segment RT. She was led to a faulty inference because the static view was interpreted so that RT was seen as an edge of the icosahedron. Her explanation was that segment US went all the way across the three-dimensional figure (solid) but segment RT went by the edge of the solid. The explanation revealed that her incorrect judgement came from the wrong visualization of the planar representation of the solid. Mary was at a higher thinking level than many students in her class, but she still visualized RT as an edge of the figure before she was able to relate the visual and logical structures. We can see from here the sharp difference between the static two-dimensional representation of a solid and the solid itself, and the strong influence of the difference on how students interpret the visual.

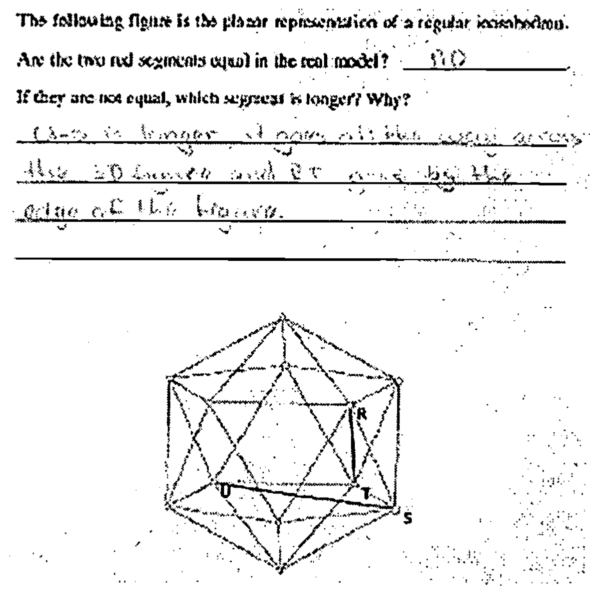


Figure 7. Mary's answer sheet for the paper-and-pencil test.

After Mary completed the paper-and-pencil test, the investigator showed her a GSP dynamic two-dimensional representation displayed in Figure 8. Mary clicked the Rotation button, and found immediately a quite different visualization from that in the paper-and-pencil situation. In order to see all possible orientations given by the dynamic representation, she chose to drag the free point on the small circle to observe the rotation in a controlled speed. The feedback that the dynamic

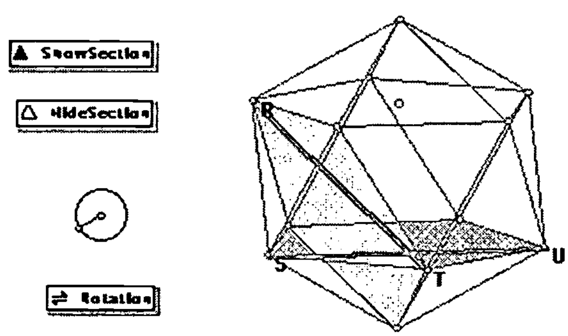
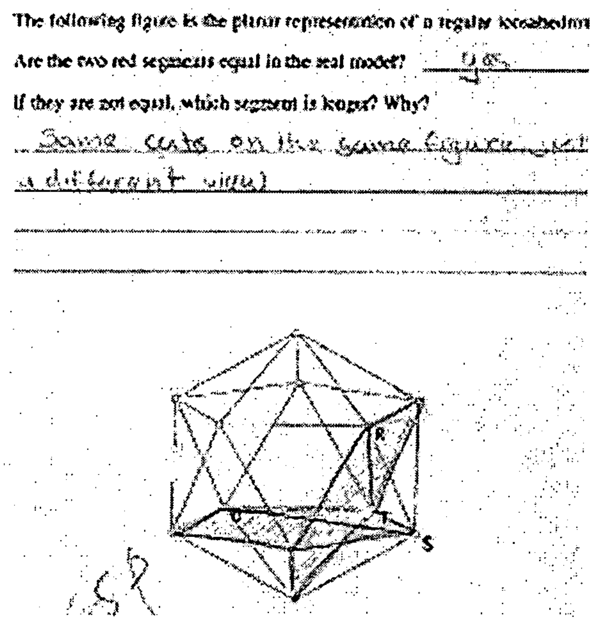


Figure 8. GSP dynamic two-dimensional representation of the problem situation.


The following figure is the planar representation of a regular icosahedron

Are the two red segments equal in the real model? yes

If they are not equal, which segment is longer? Why?

Same cuts on the same figure, just a different view)

Figure 9. Mary's answer sheet in the GSP version of the test.

representation provided allowed her to realize the congruence of segments US and RT. This time, she was able to visualize segments US and RT as diagonals of two pentagonal cross sections of the solid. Even though she did not mention this explicitly in her explanation, she did indicate the new visualization by shading these two pentagonal cross sections in the figure (Figure 9). Her idea was clearly that the two pentagons were congruent, and hence the two corresponding diagonals were congruent. This is an indicator that she used logic properties to explain her new finding and achieved conceptual understanding of the relevant spatial relationships.

The goal of enhancing the subjects' geometric thought by at least one van Hiele level was basically achieved.

Through the teaching experiments of the research, all subjects made significant progress. We used modified versions of the van Hiele level test developed by Mayberry (1983) to measure the subjects' levels of geometric thought at the beginning of the studies and at the end of each phase. Of these tests, the questions on two-dimensional shapes were from Mayberry's instrument, while those on three-dimensional objects and the criteria for scoring were created by us following the structure of her instrument. By the end of Phase 2, all subjects' geometric thought was found to be improved with the increase of at least one van Hiele level. By the end of the research, all four subjects who participated in the teaching interviews in Phase 3 and Phase 4 improved their geometric thought by at least two van Hiele levels. Two of them (Manuel and Pedro) were able to do logical reasoning for the three-dimensional tasks, reaching the Deduction level (van Hiele level 4). For example, Manuel showed what we consider van Hiele level 4 thought in a variety of tasks. In the task shown in Figure 10, the front of an Archimedian solid was provided on paper. This solid is characterized as being composed of a pair of squares and a pair of equilateral triangles at each vertex. Manuel was asked to draw (on paper) a new solid from which this solid could be sliced. This task was quite challenging because it required considerable knowledge of Platonic and Archimedian solids, as well as strong three-dimensional visualization. Manuel easily completed the diagram (of the Archimedian solid) by drawing the hidden parts of the solid. He was then able to complete drawings of both a cube from which this solid could be sliced and an octahedron from which it could be sliced.

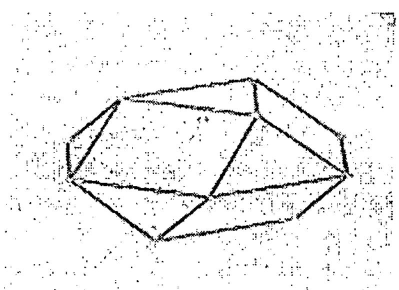


Figure 10. A sketch of an Archimedian solid characterized by the completion of the backside outlines.

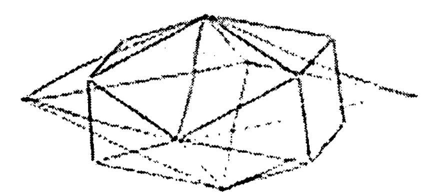


Figure 11. The sketch of one of two Platonic solids from which the Archimedian solid is "sliced."

Figure 11shows the octahedron that he drew . This indicates that Manuel was easily able to use the two-dimensional representations of the octahedron and the cube to produce the Archimedian solid. This task represents an example of complex problems that require mental processing equivalent in thought to that of a geometric proof. Sufficient GSP experience effectively helped him develop solid conceptual understanding so that he was able to solve challenging problems with or without referring to GSP representations. We believe that with consistent performance on such tasks, the subject was performing at level 4 in van Hiele's scheme.

Another example is the following task assigned to Pedro: Construct an Archimedian solid that is defined by the characteristic of having three squares and one equilateral triangle at each vertex. Pedro responded to this challenge by first drawing what he conjectured the solid would look like (see Figure 12). He then proceeded to create

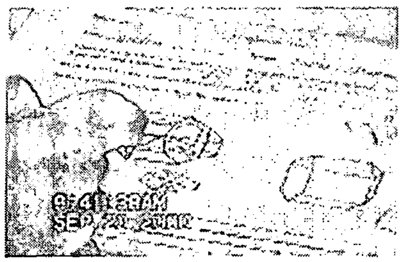


Figure 12. A drawing of the Archimedian solid that has three squares and an equilateral triangle at each vertex.

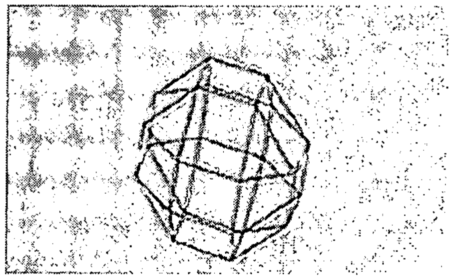


Figure 13. A dynamic GSP version of the Archimedian solid.

a dynamic GSP version of the solid shown in Figure 13. The complexity of the task was substantial. Pedro started with a "local definition" of the solid and proceeded to a paper and pencil drawing. The drawing carried with it analysis and detail so that Pedro was able to discern the structure of the solid. He was then able to construct a dynamic GSP sketch with capabilities of rotation and creation of cross sections. This seems to be indicative of the nature of deductive reasoning, and leads us to believe that this student was functioning at level 4.

Dixon (1997) found in her study that GSP instructional environment was not more effective than the traditional environment at improving students' three-dimensional visualization. However she suggested that the (test) result might have been different if the three-dimensional visualization had been measured using manipulation of blocks or the simulation of three-dimensional behavior using the computer. The results of these studies supported her conjecture.

Conclusion

Three-dimensional visualization is a complicated cognitive procedure in which the subjects need to process the information perceived from external representation and the information retrieved from internal representation in an interwoven, integrative, and dynamic manner. During the studies reported by this article, the subjects actively participated in the explorations of three-dimensional visualization in the GSP dynamic instructional environment. Through interacting with the dynamic environment the subjects had many opportunities to construct their understandings of spatial structures and spatial relationships of three-dimensional objects. The way that the GSP environment differs from the traditional environment is that the GSP environment can provide students with dynamic two-dimensional representations of three-dimensional objects, powerful transformation tools, and some other useful features such as animation and the use of buttons. Specifically, the GSP environment helps students concentrate on the logical rather than visual properties of the three-dimensional objects. The immediate feedback provided by the dynamic environment allows students to verify or change their conjectures. At the end of the studies, the subjects had progressed significantly in terms of their van Hiele levels of geometric thought. In summary, GSP and the associated activities had a distinct positive affect on the subjects when they were developing three-dimensional visualization and pursuing conceptual understanding of geometry content.

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